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Active, bodily experience Mathematics learning activities. Practices and beliefs of Italian and Australian teachers

Research Framework

The relevance of perception and bodily movement for the exploration and construction of mathematical concepts is a central topic in much of the relevant literature in education, particularly in mathematics education. The roots of this tradition can be traced all the way back to the early 1900s, to the well-known Italian contributions of Maria Montessori (1934a, 1934b) and Emma Castelnuovo (1963), as well as to Jean Piaget (1952, 1953, 1960), John Dewey (1933,1938), and Jerome S. Bruner's (1966) theoretical works.

In the last three decades, the relevance of perceptual-motor aspects in mathematical learning processes has been evidenced by studies in experimental psychology (e.g., Carlson et al, 2007; Goldin-Meadows, 2005) and cognitive neuroscience (e.g., Dehaene, 2010; Nemirovsky & Borba, 2003; Looi et al., 2016; Seitz, 2000), thanks, in particular, to studies that have emphasized the *sensuous* and multimodal character of knowledge and learning processes (Rizzolatti et al., 1997; Gallese & Lakoff, 2005). Its relevance has been even more emphasized by theories from the cognitive psychology of *embodied and embedded cognition*, developed in the two pioneering works of Varela et al. (1991) and Lakoff and Núñez (2000).

More specifically, the theories of *embodied cognition* highlight the relevance of involving the sensorimotor apparatus for the development of cognition. These theories maintain that cognition and higher thinking should not be considered confined to the mind, but instead distributed throughout the entire body (Barsalou 2008; Lakoff & Johnson, 1999). Such argument carries with it two fundamental consequences for the teaching and learning of mathematics. The first one concerns the importance of promoting activities that stimulate perception and movement in the learning processes of the discipline. The second one regards the need to consider aspects that belong to the non-verbal language, such as gestures and perception, which play a fundamental role in teaching-learning processes, both in terms of communicative characteristics (Cook et al., 2012; Alibali et al., 2014), and the production of mathematical meanings (Cook et al, 2008; Alibali & Nathan, 2012; Cook, 2018). According to Chatelet (2000), gestures and metaphors are the way to transform the body's disciplined motility into signs. Moreover, as emphasized by J.A. Seitz, *"we do not simply inhabit our bodies, we literally use them to think with"* (Seitz, 2000). For instance, recently, research groups working on mathematics education have analysed the role of gestures and nonverbal language in teaching-learning processes, which do not represent simple communicative elements, but rather substantial aspects of the development of thinking (McNeil, 1992; Valenzeno et al, 2003; Goldin-Meadow, 2012; Rueckert et al., 2017, Congdon et al., 2017). One example is the *multimodal approach*, which analysed these aspects from a socio-constructivist perspective of learning (e.g., Arzarello & Robutti, 2009; Radford et al., 2017). Other studies were focused specifically on the inseparable nature of imagination in mathematics and the perceptual-motor aspects (Nemirovsky & Ferrara, 2009), on the role of gestures (Goldin-Meadow & Singer, 2003; Carlson et al., 2007; Edwards et al., 2014; Vale & Barbosa, 2017) or on *sensuous* cognition (Radford, 2013; 2014). *Embedded cognition* theories, however, emphasize how cognition is realized, and constrained, by the mutual interactions between the body and the environment. This highlights to what extent external artefacts and cognitive processes are deeply interdependent, and how learning processes and their effectiveness, depend on the coordination between students' bodily and environmental resources (Clark 2008; Pouw et al. 2014). Interesting studies in this direction, based on theoretical perspectives such as *inclusive materialism* (de Freitas & Sinclair, 2012; 2014), have also been conducted through the use of digital technologies and resources (e.g., Baccaglini-Frank et al. 2020, Ferrara & Ferrari 2020; Shvarts & Abrahamson, 2019).

These aspects are central in the *enactivist pedagogy* as well. *Enactivist pedagogy*, places its origins in *learning by doing,* theorized by John Dewey (1916), which found ample space in the *genetic epistemology* of Jean Piaget (1896-1980), according to whom, the ability to learn depends on our ability to absorb the world through meaningful engagement. Although Piaget's work has been criticized and superseded in its component concerning the order and progression of cognitive development that he theorized (Spencer & Darvizeh, 1981; Wallace et al. 1987), undoubtedly the perspective regarding the importance of the learner physical interaction with representations has instead found a fortunate following. Such a perspective was certainly embraced by Jerome S. Bruner (1915-2016), who was the first to use the term *enactive*, theorizing learning as situated on a continuum between concrete and abstract, through the transition from enactive representations (such as manipulation) to iconic representations (such as pictures) to symbolic representations (such as words) (Bruner, 1966). The *enactivist* perspective on *embodied cognition* finds its origin in the contribution of Varela (1991), which emerged from Merleau-Ponty's phenomenological perspective (Merleu-Ponty, 2013). Researchers Abrahamson, Dutton and Bakker (2022) theorize an *enactivist pedagogy of mathematics* in the article *Towards an Enactivist Mathematics Pedagogy*, which provides a manifesto of this disciplinary pedagogical philosophy. The *enactivist* pedagogy focuses on the materiality of mathematical thinking, in opposition to commonly proposed teaching practices that ignore the *"psychological experience of thinking […] by denuding concepts of their corporeality"* (*ibidem*, p.157), that is, understating the relationship between body (movement and perception) and mind. According to Abrahamson et al., learning mathematics has a bodily and motor origin and is developed through conscious discourse involving descriptive processes of measurement, analysis, modelling, and symbolization, through which perceptual structures are transformed into mathematical entities that retain an active role. Therefore, how we make learning accessible, according to the researchers, should derive from thinking about how to enable students to actively experience mathematical concepts by designing an environment, artefacts, and assessment, aligned with this purpose.

Other perspectives considered in mathematics education with regards, in particular, to the role of body movement, come from research conducted by psychologists and philosophers such as Maxine

Sheets-Johnstone (2011). In her work, she introduces the idea of *thinking in motion*, focusing on the kinaesthetic aspect of cognition, considering movement as both a means of thinking and of reproducing thought. Earlier, a focus on movement, more properly on the proprioceptive and kinaesthetic aspects of perception, was described by the physiologist of perception and action Alain Berthoz in the book *Le sens du mouvement* (1997). This book focuses on mathematics, and, more specifically, when addressing geometry, Berthoz explicitly refers both to the mathematical philosopher Chatelet (2000), who inspired many of the studies mentioned (e.g. those on gestures), as well as to the philosopher Poincarè and his statement, contained in the book *Le science et l'Hypothèse* (1908), which declares geometry as originating from the body and its actions. The philosopher of mathematics Giuseppe Longo (2005) also supports a similar position. In the following years, the discussion around research on the role of bodily movements in mathematics was not only considered from a cognitive point of view, but also centred around the epistemology of the discipline itself (Núñez, 2006). Indeed, from the reflections of these authors, the constitutive role that body movement plays for mathematics itself, both in concept and practice, has emerged. However, it must be pointed out that this aspect has been greatly underestimated throughout the last century in the Western tradition of mathematics education. This is evident, for example, from the presence of many Bourbakist definitions in mathematics textbooks (Munson, 2010), such as the ones referring to functions (Denbel, 2015).

A further area that is important to consider because of its profound entanglement with this theme is the use of manipulative materials and tools for teaching and learning mathematics. Many of the theoretical frameworks and studies presented so far have led to the development of educational materials and pathways for teaching-learning (e.g, Bussi & Maschietto, 2006; Baccaglini-Frank & Maracci, 2015; Baccaglini-Frank, 2015; Bussi et al., 2018; Carotenuto et al., 2020). Other theoretical perspectives, such as the Semiotic Mediation (Bussi & Mariotti, 2008) have explicitly studied these activities for the development of mathematical thinking. Specific studies have investigated the characteristics of manipulative representations (Belenky & Schalk 2014; Carbonneau et al., 2013), as well as levels of instructional guidance that made use of manipulatives (Marley & Carbonneau, 2014) or teachers' beliefs and practices toward their proposal in school (Carbonneau & Marley, 2015; Golafshani, 2013; Vizzi, 2016; Puchner et al., 2008).

As briefly introduced above, the research regarding the centrality of the body perception and movement in mathematics has a long and extensively debated tradition. Furthermore, in recent decades, the role of students' active, bodily experience in the exploration and construction of mathematical concepts has attained increasing attention in the research on mathematics education. As pointed out by Drijvers in the ERME-11 plenary (2019), the growing interest in research from this perspective is evidenced, for example, by the fact that two special issues of *Educational Studies in Mathematics*[1], have been devoted to *embodiment* in mathematics education. Furthermore, several theories focused on perceptual-motor involvement in the mathematics teaching-learning process have been proposed. Encouraged by experimental findings, research embracing theoretical perspectives that highlight the centrality of students' bodily perception and movement has also recently developed a wide range of innovative educational artefacts and proposals.

However, this growing interest in research and developments on a theoretical level have not been matched by an equally ample resonance in classroom practice. The teaching of mathematics in schools, as pointed out by the OECD's international surveys (2009; 2016), is often far from these perspectives, being instead still largely anchored to a transmissive teaching approach. Dominant teaching practices, focused on strategies that aim at, for example, *clarity of instruction* (OECD, 2019), are often geared towards teaching procedural mathematics rather than promoting cognitive activation,

[1] N.57(3) Published in 2004, N. 70(2) published in 2009

usually supported instead by exploratory, active learning, and problem-solving practices. Thus, the presence of the aforementioned perspective in teaching practice is uneven. This gap between research findings and the uneven proposal of these activities in classrooms warrants a research interest in the implementation in school of active, bodily experience mathematics learning activities.

Research goals

As we have briefly outlined, over the years, many research findings, both experimental and theoretical, have emphasized, on one hand, the importance of actively engaging students in experiential activities and, on the other hand, the role played by perception and movement in mathematics teaching-learning processes. National and international educational policies have adopted, to a greater or lesser extent, research findings, implementing them according to the mathematical tradition and the culture of the specific contexts where teaching takes place. However, we do not possess adequate information concerning the nature and scope of educational proposals in schools that are aligned with what is indicated by research in this regard, other than the recognition of an uneven diffusion, for example, in Italy (Bartolini Bussi et al., 2010). In order to shed light on the gulf between research and school practice, it could be useful to investigate the actual design and implementation in mathematics classes of learning activities in which students are actively involved through their bodies and movement. In this way, it could be possible to describe the scope and nature of these proposals at school, including possible adaptations and omissions of key components, and to explore the possible presence of contextual factors that may support or inhibit such activities (Century & Cassata, 2016).

The variety of theories mentioned in the previous section corresponds to constructs resulting from the specific philosophical, psychological, and pedagogical roots. Analysing the implementation with an exploratory study aimed at understanding teachers' perspectives, the research needs to be based on a negotiation of meanings with relevant actors. Setting aside the theoretical differences, we identified a comprehensive construct, an overarching framework for the multitude of theoretical proposals developed, that could be clear and easily accessible to teachers, to be the object of our research on implementation. Thus, the terminology active, bodily experience mathematics learning activities, abbreviated hereafter in ABM activities, refers to activities designed according to the perspective of enactive-embodied learning or, more generally, to activities in which students are actively engaged in exploring mathematical concepts using manipulatives, tools (virtual or physical), or whole-body movements. Two main components are encapsulated in this construct: the students' active engagement in mathematical exploration and their perceptual motor involvement.

The present study explores the implementation of ABM activities at school (Century & Cassata, 2014), with a specific focus on teachers' views and beliefs. We will observe implementation from the perspective of teachers, assuming that they can give us precious insights on the current implementation as conducted in classrooms. The teacher's beliefs and experience play a significant role in educational change (Coburn & Talbert, 2006; Peterson, 2013), and their opinions on ABM activities could thus have a drastic impact on their implementation (Domitrovich et al., 2008; Ruiz-Primo, 2006), as highlighted in the specific case of the introduction of manipulatives by Golafshani (2013) and Vizzi (2013).

The inquiry is carried out in Italy and Australia in order to explore the presence of latent variables and implicit characteristics linked to the specific cultural setting, which might not emerge by conducting the survey only in a single educational system.

This study investigates the proposal and implementation of *active, bodily experience mathematics learning activities* in Italian and Australian schools. Given the multiform and complex nature of the operational construct under investigation, it has been essential to clarify its components and to shape its attributes, possible declinations, and adaptations in different contexts. Therefore, the first objective of the research was to identify the elements that can characterize ABM activities and their implementation. To this end, in addition to reviewing the direction of research findings and official guidelines at a national and international levels, we conducted an exploratory study with academics in the field of mathematics education. Indeed, they hold a privileged position to pursue such a goal because, from a research perspective, they are in continuous dialogue with school contexts. Experts' opinions help recognise the core elements and expected outcomes of the ABM activities, and to classify determinant factors in and for their implementation.

Specific research questions to achieve this first aim were as follows:

- ***RQ_1a***. *From experts' perspective, how are ABM activities conceptualized and characterized in Italy and Australia?*
- ***RQ_2a***. *What are teacher's characteristics (knowledge, beliefs, awareness) that should come along with ABM activities implementation in school?*
- ***RQ_3a***. *What are possible hindering / facilitating factors for ABM activities implementation?*

The research also explored the perspectives of both primary and secondary school mathematics teachers, with respect to the implementation in schools of ABM activities and the proposal in their teaching daily practice. This study is aimed at identifying factors that support or inhibit the implementation of ABM activities, inferring the possible relation between teachers' beliefs and their disposition to implement (or the current implementation of) ABM activities, as well as other influential teachers' characteristics or contextual factors. Furthermore, we sought to gain insights for the effective implementation of ABM activities surveying the current implementation, particularly identifying the factors that affect the practice, the existence of different models of practice, and classifying teaching profiles and characteristics that determine their teaching effectiveness.

The specific research questions that guided the inquiry have been as follows:

- ***RQ_1b.*** *What are teachers' beliefs (in terms of expected outcomes, limitations, difficulties, constraints identified) about ABM activities and possible/current implementation in daily practice?*
- ***RQ_2b.*** *Are there teaching profiles (educational background, teaching experience, beliefs on mathematics teaching and learning) or external characteristics (culture and characteristics of the school context, curriculum and directives on educational policies, research efforts) that may determine the readiness to implement ABM activities in school?*
- ***RQ_3b.*** *When ABM activities are implemented, to what extent are their proposal and accompanying teaching profile aligned with the indications provided by research findings and academic experts?*

Any cultural differences within the two different contexts are also analysed.

Research design and methodology

The research is an exploratory mixed-method study on the implementation of ABM activities in Italian and Australian mathematics classrooms, and associated teaching practices.

Research design included:

- a desk audit of relevant research literature (theoretical perspectives and empirical research focusing on the involvement of the body and movement in mathematics learning activities, consistent with an active, experimental, and hands-on approach), and relevant national and international curricular documents, guidelines, and educational policies to identify the role of the body and movement in mathematics teaching and learning.

- semi-structured online interviews with academic experts on mathematics education aimed at documenting experts' views on ABM activities to identify a conceptual framework on the main issues outlined in the teachers' survey. Analysis of the narrative material has produced a conceptual framework that highlights the broader opinions, as well as those of various experts', on core elements, as well as what the expected outcomes of these activities are, and the factors believed to be determinant in and for their implementation.
- a survey addressed to primary and secondary teachers, which consist of:
  - an online questionnaire focused on teachers' beliefs and practices regarding teaching and learning mathematics in general, and particularly to ABM activities. The web-based instrument combines rating items, multiple-choice items, two vignette-items, and a few short open-ended questions, differentiating primary and secondary school teachers using filter questions. After completing the questionnaire, teachers interested in participating in an individual interview were asked to provide their email (in the form) for further potential contact by the researchers.
  - individual semi-structured interviews in Australia / focus group in Italy with a restricted number of respondents aimed at: providing greater insight into issues raised in the participants' survey responses; and at delving further into some topics for which the questionnaire might not yield sufficient information.

Experts' perspectives

Academics in mathematics education represent a *trait d'union* between research and school. Indeed, through experimental research conducted in the classroom and professional development courses for teachers and prospective teachers, they participate in the world of research by addressing the schools and their students. Thus, they play a significant role in identifying what directions should research take when analysing and interpreting the implementation of research findings in schools.

First, interviews with academics complete the literature review to provide a characterization of ABM activities by defining their essential elements, expected outcomes, founding principles, and possible different interpretations. As we have already pointed out, ABM activities represent an operational definition, constructed for exploratory purposes, including a variety of theoretical perspectives that can contain significant differences too. It is therefore essential to identify the principles that characterize them as the object of study. In doing so, the experts' inputs were pivotal references for envisaging connections and gaps between research directions and teachers' perspectives.

Secondly, their participation allowed a contextual characterization, that is, to describe the features of the specific teaching culture considered. This translates into identifying possible examples consistent with the object of study and a terminology for ABM activities familiar and easily recognizable by teachers. These communicative and linguistic choices proved to be crucial in designing the survey instruments aimed at teachers. Moreover, academics' involvement allowed to draw interpretive lines on possible differences in implementation related to the specific structural characteristics of the school system and the culture, both mathematical and educational, in which students and teachers are immersed.

Participants

The experts in mathematics education who participated in the project were selected based on experience alongside teachers and for their research interests, which were akin to our research topic. The six experts in Australia are academics, belonging to MERGA (Mathematics Education Research Group of Australasia). Three of them are former secondary school teachers, with expertise in professional development courses aimed at mathematics teachers, experimenting in school teaching innovations. Their research interests range from initial and professional teacher education to inquiry-based learning; from the use of technology in mathematics education to mathematical Literacy and Numeracy; and from implementing teaching innovations in elementary school to reforms in curriculum and assessment. They all have experience in teacher education and international research.

In Italy, we have selected and interviewed 9 experts in mathematics education. They are 7 accomplished academics and a teacher-researcher, who have a wide range of different research interests: mathematics difficulties and the use of representations, teachers' beliefs and problem-solving, teacher education, semiotic mediation, proof and argumentation, cultural transposition, multimodal approaches and gestures, Montessori method education. Overall, all of them have long experience in teachers' professional development courses and empirical research in classrooms and they are familiar with the topic. Furthermore, seven of them are memebers of the National association of research in Mathematics Education AIRDM (Associazione Italiana di Ricerca in Didattica della Matematica).

The academic experts were recruited through an email invitation asking them to contribute to the research, and they joined the project voluntarily. After filling out the informed consent form for the interviews, we proceeded to schedule the interviews, which were conducted via Zoom, in the period between May 2021 and December 2021 in Italy, between November 2021 and December 2021 in Australia.

The interview protocol

In order to collect the academic experts' opinions, we conducted individual semi-structured interviews via Zoom, approximately one hour-long. The interview prompts were designed to assess the experts' views on key aspects of implementing ABM activities at school, especially in relation to teaching practices. The first goal of the interviews was to gather the researchers' opinions on the proper terminology to define the activities under investigation in a clear and accessible way for teachers. This exploratory phase should also provide a set of examples that might be commonly known and recognized by teachers, at different school levels. Furthermore, the interviews helped shape a conceptual framework of academic experts' views around the main questions underpinning the survey on teachers' perspectives. The prompts of the interviews are listed in the box below (Tab.1).

I	*Whether and why is it important to implement ABM activities at school?*
II	*What are the beliefs that should guide teachers in proposing them?*
III	*Which levels of awareness and knowledge should accompany teaching when implementing ABM activities? (e.g., in terms of teaching strategies, assessment, etc.)*
IV	*Which characteristics concerning the implementation of ABM activities at school determine their teaching effectiveness?*
V	*What are the main limitations of the use of these activities in daily teaching practice?*

	What are factors that could hinder/favour the implementation of these activities at school?

Table 1. Prompts for mathematics education academic experts' interviews.

Data analysis

Transcriptions were done manually, by making use of the slow speech dictation mode provided by the free software for audio file playback *Listen N Write Free*. Transcriptions are in *Jeffersonian* simplified style (Jefferson, 2004) (Tab.2) to report as faithfully as possible the narrative from the interviewees, including emphases and uncertainties (Sacks et al., 1974). The whole transcriptions of academics' interviews can be found in Appendix 3.

Jeffersonian Simplified system of transcription	
[	Speech overlay
>text<	Accelerated speech
<text>	Slow speech
text	Emphasized word
wo:rd	: for sound lengthening
(0.5)	The number inside the bracket indicates the duration of a pause in seconds and tenths of a second
((action))	Double brackets indicate a description of an action
( )	A part of speech that is not understandable
=	Words spoken stuck together

Table 2. Legend of symbols for transcription (Jeffersonian simplified system)

The narrative material thus transcribed was imported for analysis into *MAXQDA Software* (*Analytics Pro 2022 version*). Interviews were analysed according to the *Qualitative Content Analysis* method, using inductive category formation procedure (Mayring, 2015). The so-called *open coding* in Grounded Theory (Cohen et al., 2017), was used in the first instance to refine the results with a focused, axial coding (Strauss, 1987; Ezzy, 2002). Diagrams were used in the analysis process to help thinking on a conceptual level, showing the relationships between concepts (Corbin and Strauss, 2008). In addition, concept maps (Daley, 2004) were used as a tool to represent the conceptual framework of the academic experts' perspective that emerged from the data gathered on each theme.

The first two questions, about the terminology and examples, where categorised using both a data-driven categorisation proposed by one researcher (Expert 4), and external criteria. For instance, terminology with the same linguistic root were grouped. The examples provided were categorised in the same groups according to the following criteria: if they were in the same area of contents, if they referred to the same school level, if students' physical involvement (e.g., whole body movement or handling) and tools used were of the same typology (e.g., virtual or digital).

The analysis of the other questions (illustrated above in Tab.1) produced a system of categories and subcategories, briefly illustrated in the table below (Tab.3). The main themes addressed by the interviewees in response to the protocol questions were coded and grouped in categories. The system of codes and sub codes generated represent the core of the analysis and of the interpretation of the results. It was conducted by making use of concept maps for each macro-category, in which the nodes consist of the codes' labels. Each code represents a *natural unit of meaning*, that is, a relevant theme that emerged in the narratives, and the codes are organized according to the categories and subcategories. For instance, within the category *Influential factors*, in the sub-category *Ambivalent factors,* we can find the code *FA1* with the label "*Availability of spaces and resources in the school*". This system of categories and sub-categories was created following criteria of *similarity or difference* (e.g., the set of opinions with respect to the same variable, such as the assessment of ABM activities when asking for instructional strategies), *concordance and opposition* (e.g., hindering or facilitating factors for the implementation of ABM activities in the classroom) or *inclusion* (this is the case of the creation of sub-categories from an original category) (Trinchero, 2002). In the procedure, we adopted a hermeneutic approach to refine the system of emerging categories and codes, on the basis of criteria of interpretive clarity and informational accuracy by rereading and cross analysing the narrative materials in several cycles.

CATEGORIES	SUBCATEGORIES	
IMPORTANCE (I)	Justificatory reasons (IG)	
	Operational reasons (IO)	
CHARACTERISTICS I (of the teacher) that should accompany the implementation	Beliefs (CCv)	To benefit from the introduction of the activities (CCvB)
		On mathematics teaching and learning (CCvM)
	Awareness (CCp)	On the specific activities (CcpA)
		On mathematics teaching and learning (CCpM)
	Knowledges (CCs)	
LIMITATIONS (L)	Inherent in activities (LI)	
	Due to implementation errors (LE)	
INFLUENTIAL FACTORS (F)	Ambivalent factors (FA)	
	Inhibiting factors (FO)	External (contextual) factors (FOE)
		Internal (teacher) factors (FOI)
	Facilitating factors (FF)	External factors (FFE) (which could facilitate the introduction)
		Factors that might affect teacher attitudes (FFC)
STRATEGIES FOR THE EFFECTIVENESS (SE)	For introduction (SEIn)	
	For effective implementation (SEIm)	The selection of the activity (SEImS)
		Students and classroom management (SEImG)
		The assessment (SEImV)
		The role of the teacher and teaching strategies (SEImI)
		The significance of the activity in teaching program (SEImO)

Table 3. System of categories from the analysis of academic experts interviews

Reliability of the Data-analysis process

Proceeding hermeneutically could ensure the *stability* of the analysis, however, it does not guarantee that the codes assigned to the text units will be reassigned in the same way by another independent coder. In order to get a measure of the trustworthiness, a fairly common method in educational research is to make use of *investigator triangulation* (Denzin, 2009). It consists of involving one or more external coders to triangulate the data analysis system, checking trustworthiness in terms of *inter-rater reliability* (Krippendorff, 2004) on a representative sample of narrative material. In the research presented here, a *reliability process* was planned, as outlined by Syed and Nelson (2015). We involved two external researchers in a refinement phase, for partial analyses of significant coding patterns, and two coders whose task was to conduct a final validation of i*nter-rater agreement*. According to Geisler and Swarts (2019), the protocols analysed by the two external coders correspond to 20 percent of the total narrative material. We report below the two accuracy indexes, which measure the degree of agreement of each external coder with respect to the coding performed, and the index of agreement (*inter-rater* or *intercoder agreement*) of the coding performed by the two external coders, thus providing a measure of the reliability of the analysis. Usually, it is believed that a good agreement index should reach at least 80 %; the closer to total coincidence in coding (100%) the more reliable the coding is considered to be (Mantovani & Kanizsa, 1998). In our case the percentile agreement index shows that the analysis is sufficiently reliable:

$$i_{agreements(Cod.1,\, Cod.2)} = \frac{100\times\ agreements_{(Cod.1,\, Cod.2)}}{agreements_{(Cod.1,\, Cod.2)} + disagreements_{(Cod.1,\, Cod.2)}} = 83\%.$$

In addition, the accuracy indices

$$i_{accuracy\ (Cod.N)} = \frac{100\times\ agreements_{(Cod.N,\, Code\ System)}}{agreements_{(Cod.N,\, Code\ System)} + disagreements_{(Cod.N,\, Code\ System)}}$$

of both external coders are quite good, respectively $i_{accuracy\ (Cod.1)} = 85\%$, $i_{accuracy\ (Cod.2)} = 96\%$.

Another issue for the reliability of the analysis method concerns the researcher's autonomy in creating the coding system, which could then be compromised by the presence of predetermined assumptions (Trinchero, 2002). We attempted to limit this effect by inductively eliciting analysis from text units and reverting to coding narrative texts based on code and category systems, questioning fidelity and interpretive clarity reshaping coding systems until no more inconsistencies or ambiguities were found. Finally, to further ensure reliability, we returned the entire interview transcript to each academic expert interviewee.

Teachers' perspectives

Teachers were directly involved in the research through the completion of a questionnaire and follow-up interviews. The questionnaire is a web-based tool created with Qualtrics software, in the updated version available to ACU students and researchers. The estimated time for filling out the questionnaire is around 20 minutes. After completing the questionnaire, teachers are asked if they are willing to participate in a follow-up online interview. In Italy, these are focus groups that gather teachers from the same school levels, in Australia they are individual interviews.

Design

The survey items cover dimensions derived from the literature concerning teachers' beliefs on mathematics teaching and learning (Beswick, 2012; Van Zoest et al., 1994; Dionne, 1993; Ernest, 1989), conceptions of educational material usage (Skoumios & Skoumpourdi, 2021), and beliefs and instructional practices with manipulatives (Carbonneau & Marley, 2015; Golafshani, 2013; Vizzi, 2016). Other items were adapted from items on existing surveys such as OECD TALIS 2018[2] and IEA TIMSS 2019[3]. There will be additional items concerning new explorative dimensions specific to our research interests.

There are two parallel versions of the survey, one for primary school teachers and one for secondary school teachers, with minor adaptations to suit the teaching context. The survey consists of five sections:

1. *The School* – concerns general information about the current school (e.g., government / non-government school; traditional / school based on a specific educational method such as Montessori) and school level/s that the respondent is currently teaching.
2. *General* – designed to provide information about the teacher's educational background and

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[2] OECD (2019). *TALIS 2018 Results (Volume I): Teachers and School Leaders as Lifelong Learners*. OECD Publishing.

[3] https://www.iea.nl/studies/iea/timss

teaching experience.

3. *Beliefs* (a) – including broad beliefs about teaching and learning mathematics (e.g., the role of the teacher or peers in the learning process).
4. *Beliefs* (b) – specific beliefs about the ABM activities (e.g., for which school level these activities are considered appropriate, what kind of educational impact is expected to be achieved, what factors can possibly limit their use, what kind of evaluation/assessment strategy may be appropriate).
5. At the end of the fourth section, a **filter question** concerning the actual use of these activities in daily teaching practices splits the questionnaire into two alternative parts on the basis of the teacher's use of these activities in their teaching practice (Yes/No). This next section asks teachers for additional information such as the reasons for this choice, what other teaching strategies they deem to be effective, and comment about their implementation in classrooms (if used).

The survey combines Likert-type, multiple-choice, short open-response, and vignette items. The Likert-type scales are commonly involved in research that employs questionnaires to gain information on complex variables like beliefs (Nunnally, 1994), although there are some risks to be taken into account when using Likert-type items in a survey. For instance, it is difficult to know how the respondent interprets the words used in the items, and, moreover, the main risk is that "*Likert items do not carry with them good ways for determining how important the issue is to the respondent*" (Ambrose et al., 2003, p.35). Consequently, it is difficult to understand the belief system of participants (such as the centrality of certain beliefs). The few short open-ended items allow participants to express ideas without the constraints of choosing one or more alternatives as in multiple-choice items. However, in the survey, we mostly use multiple-choice items, especially when asking about teaching practices, in order to obtain responses that are aligned with the variables we want to investigate.

Two vignette items are included in the questionnaire: the first to investigate teachers' beliefs on ABM activities, and the second to gain insight on the instructional strategies teachers use when implementing them in classrooms. Regarding the empirical study of beliefs, which has always presented difficult methodological questions for researchers (Finch, 1987), the use of a vignette could be a valid technique to be used. The criticism often levelled at questionnaires or interviews is that *"these techniques pose vague questions to respondents, who consequently answer in terms of their own mental picture of the task. Therefore, the information derived from such data is non-referenced, and cannot be standardised across all the respondents."* In this sense, vignette items are a good solution to reduce this misleading effect, because they present respondents *"with a more concrete and unambiguous stimulus to refer to"* (Poulou, 2001, pp.51-52). Research in mathematics education highlighted that vignette items are also a good instrument to be used to get information about instructional practice (Stecher, 2006). However, we have to take into account the possible inconsistencies that might exist regarding participants' descriptions of what they would do at an abstract level and what occurs in reality. Indeed, as Skilling and Stylianides have underlined, when using vignette items, it is important to be aware of *"participants' espoused beliefs are representative of their intentions rather than predictions of behaviours and are valuable for providing insights into their interpretations and perceptions"* (Skilling & Stylianides, 2020; p.554). Nevertheless, since our aim is to investigate teachers' perspectives, instead of teachers' behaviours, this fact is not a major limitation. The vignette items for the questionnaire were designed upon the framework of Skilling & Stylianides (2020).

Data Analysis

Descriptive statistics (e.g., frequencies, percentages, cross-tabulations with Chi-squared test) and correlations resulting from recording the similarities and the differences among the basic variables of the sample have been used to analyse the Likert-type and multiple-choice item responses. Open-ended questions were initially coded following an analytic induction from the content (Cohen et al., 2017; e.g., in Hourigan et al., 2016). Then the initial codes have been grouped into categories or themes, which have been examined for patterns across school levels. The number of comments from teachers at each school level in each of the broad categories have been counted to provide an indication of the relative emphasis on each category/theme across school levels, to identify the main trends and recurring themes.

Dissemination strategies and participants

Australia

In Australia primary and secondary schools' mathematics teachers were recruited from across Australia, via National and State mathematics teacher professional associations' Facebook pages/groups, or associations' newsletters. This entailed posting an anonymous link to the questionnaire on Facebook pages/groups/newsletters with a request for it to be completed by participants. By clicking on the link, potential participants gained access to information about the project. Participants had then been required to provide consent prior to completing the questionnaire. In addition to advertising via Australian mathematics associations' Facebook organisations newsletters, we sought to advertise through umbrella organisations, and broader teacher organisations bringing together Australian teachers.

The Australian sample consists of 81 respondents: 15 of them are primary school teachers and 64 are secondary school teachers. Mainly, they were working in Non-government school (45) while 29 in Government school. Most of them are expert teachers (43), 15 are middle-expertise teachers and 15 newly employed teachers.

Upon completion of the questionnaire, 16 teachers provided their email indicating that they were willing to be contacted again by the researchers to conduct an individual follow-up interview. With nine of them it was possible to arrange a 30-minute interview between April 2022 and May 2022.

Italy

The dissemination strategy in Italy was quite different: in addition to posting on teachers' Facebook groups/pages and disseminating through teachers' association mailing lists, we directly contacted most Italian schools with a direct e-mail to principals, asking them to circulate the link among mathematics teachers.

We thus reached a convenience sample of 1301 respondents: 1206 of them answered at least the first filter questions, but only 877 of them could be considered as having completed the questionnaire. The sample of respondents consists of a minority of newly employed teachers (194), who have less than 3 years' teaching experience, and medium-experienced teachers (305), who have teaching experience ranging from 4 to 10 years, while a large majority (614) consists of experienced teachers (who have accrued more than 10 years of teaching experience). Among the 1206 respondents to question Q1, 540 said they were elementary school teachers and 666 said they were secondary school teachers. Among the respondents to the third question (Q3), 1109 of them were currently teaching in a public school, while 43 of them in a private school.

Among the respondents, 292 were willing to take part in a one-hour follow-up focus group. We managed to organize 6 online focus-groups (about 9 teachers for each focus groups) between January 2022 and February 2022: 2 for primary school (grades 1-5), 2 for middle-school (grades 6-8), 2 for Secondary school (grades 9-13).

Some preliminary results

In the following paragraph we would like to provide a glimpse into the research results. Indeed, we collect a large amount of data and it is not possible to summarize them in a few lines. Furthermore, the whole research scheme we considered is quite complex: we involved two different contexts, which have two different cultural background, and, moreover, there are multiple levels within them. There is the level which is the one of experts and a second that is the teachers' one. In addition, the latter is divided into primary and secondary school teachers. Therefore, in the followings, we firstly provide an example of one of the main differences between the perspectives of Italian and Australian experts regarding the conceptualization of ABM activities, secondly, examples of some differences between primary and secondary school teachers, and finally, some features that could highlight to what extent the experts indications are aligned with the implementation.

Comparing the two contexts. A first example: different conceptualizations of the ABM activities

Considering contexts that have a culture of mathematics teaching that is dissimilar on many fronts, as Italy and Australia, allows us to consider possible differences in the characterization of ABM activities depending on the context's culture, as well as to formulate hypotheses regarding factors that may distinguish their implementation (Huang et al., 2020).

From the analysis of the academic experts' interviews, aside from some common traits, a significant cultural difference was particularly evident. While the Australian academics tended to consider the activities as a way of bringing mathematics closer to students, *"I think math could be taught in a very abstract way and if - particularly for younger children- if you want them to engage and enjoy maths I think it's gonna be practical and real, and using manipulatives just helps them to see this being something real"* (Expert 1, p.28), showing how it represents a tool for investigating and interpreting the world, e.g., *"to visualise, [..] envision mathematics in the world"* (Expert 2, p. 38-42), Italian ones related them with the possibility, for a greater number of students, to access a deeper and more *relational* understanding of mathematics (Skemp, 1976), through a meaningful construction of knowledge that also considers its history and evolution. For instance, the Italian expert Maria Mellone stressed to what extent in these activities there is "*the possibility of a more meaningful learning, where students are actually an active protagonist in the construction of them knowledge"*, *"allowing for a multifaceted approach to a mathematical meaning"* and including also "*examples that relate to the history of mathematics, because the mathematics that we know today has been mainly developed from these examples. And thus, by the way, not always consciously*", as suggest by expert Maria Giuseppina Bartolini Bussi. Furthermore, the expert Ferdinando Arzarello emphasized that in these activities clearly emerge *"what role the body plays in the solution* [of mathematics task] *and thus the multimodality with which we relate to mathematics, which is fundamental"* in particular for *"opening up the* [teaching-learning] *proposal on multiple channels and having the belief that this actually facilitates more students to follow the teacher in the construction of knowledge, that is crucial",* as suggested by Italian expert Anna Baccaglini-Frank.

The different characterization, evident throughout the interviews, clearly emerges when analysing the examples of the ABM activities proposed by the experts, both in terms of the content areas concerned and the typologies of materials and tools involved. The Italian researchers showed a greater interest in more traditional mathematical disciplines (e.g., activities from the geometrical tradition) with an emphasis mostly on the conceptual and theoretical construction of knowledge. On the other hand, the Australian academics cited many examples of mathematics modelling and real-world problems, or activities related to the area of probability and statistics, which are completely absent in the Italian context. In addition, the Australian academics quite commonly referred to examples in interdisciplinary areas, unlike Italian researchers. In the Italian context, ABM activities are instead much more often conceptualized as ends for the discipline itself. Evidence of this are the many references to the history and development of mathematical ideas that emerged repeatedly from their narratives, involving references to examples with classical tools that have characterized the evolution of mathematics (such as the abacus, the ruler and compass, or mathematical machines). Finally, the

Australian academics gave much less space to examples that recalled the use of a specific material designed for instructional purposes for the conceptual learning of mathematics, preferring materials related to everyday life and contexts. Beyond the examples, this characteristic emerges cross-categorially in the researchers' contributions. For instance, as illustrated in the conceptual map below (Fig. 1), showing the indications regarding the knowledge a teacher should possess to implement ABM activities, although most of the indications are in common, the Italian researchers stressed the importance of knowing the history and development of mathematical ideas: e.g.,

> Teachers need to know the epistemology, the philosophy and, nonetheless, the history of mathematics: how humans first came to certain concepts can be a fairly natural way to present them to children. Thus, it is necessary to know mathematics and, furthermore, some ancient mathematics (Italian expert Benedetto Scoppola, p. 63).

Meanwhile, the Australian academics highlight the need of specific knowledge to link formal mathematics to the experience of reality, as emphasized in the following contribution:

> It requires more experience in the teacher to be able to envision the mathematics in the world [...] They have to see the mathematical ideas that are at play. And I think for most teachers, both primary and secondary, they don't have that experience. So they don't yet know how to make the links. They might know the mathematics but they haven't linked it. (Australian Expert 2, p.42)

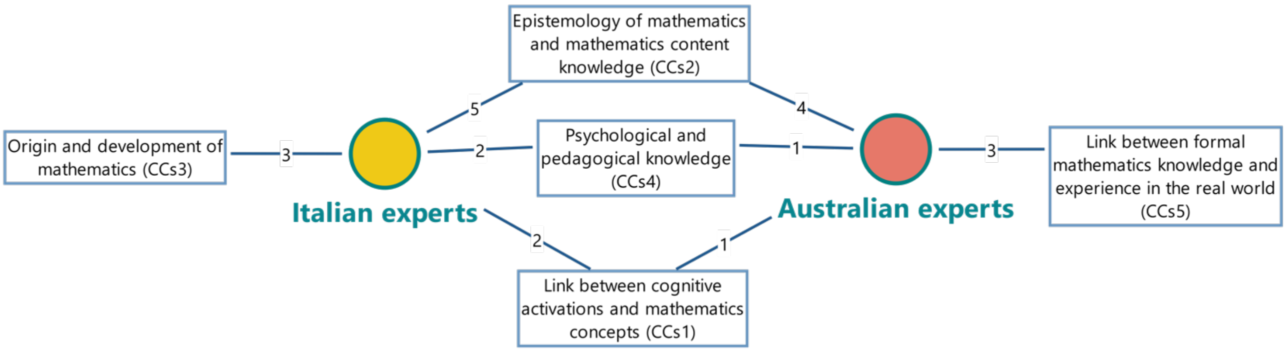

Figure 1. Concept Maps: Necessary teachers' knowledge for implementing ABM activities (XMap, MAXQDA Analytics Pro)

The teachers' survey confirmed this difference in the conceptualization of the ABM activities, although Australian teachers, especially in the follow-up interviews, called out much many geometrical examples.

Commonalities and differences between primary e secondary school teachers

The results show a tendency for greater resistance to the proposal of ABM activities in secondary school teachers than in primary school teachers. Indeed, there is a general belief that these activities may be suitable only for children in the lower grades, as reflected in the teachers' responses in agreement with the expectations of academics. Although the latter are convinced that implementing ABM activities is especially relevant for the early grades, *"the younger the learner are, the more we need to encourage and help them to do that enacting physically"* (Expert 6, p.19), they nevertheless see them as valuable learning experiences for all students, e.g. *"I think it's for all students, all students"* (Expert 4, p.32). Teachers share this belief to a lesser extent, as stated by some of them in response to question Q_14 and in some follow-up interviews, e.g., "*In the high school setting, the active body idea really makes it tokenistic. [...] I think it's more for early conceptualisation of, basic ideas, in the primary years"* (Teacher X). Indeed, this view of mathematics teaching seems to be deep-rooted in school systems:

> .. what we do in our standard school system is we say: "Right we start with concrete but we're going to come up the linear hierarchy of the curriculum and we're gonna.. You're not kids anymore so you don't need concrete, right? You know, you're going to be able to- Now you are grow up and you gonna do real maths, you know?". It's so frustrating. (Expert 3, p. 131)

In accordance with the research results, the main difficulties experienced by students are related to the so-called *transfer of learning*, that is, applying what has been learned to other contexts and in relation to formal learning. This difficulty was also pointed out by the teachers involved in the research, both primary and secondary, but it seems to have a great impact for secondary school teachers.
Among the main limitations and reasons for not implementing ABM activities, indicated by both primary and secondary school teachers, we can find the time pressure and coverage of curricular topics, the classroom management and resource availability and the school's teaching culture (principals, colleagues, parents, students). Instead, the lack of familiarity and guidance is only mentioned by secondary school teachers.

To what extent the perspectives of experts and teachers are aligned?

What seems to emerge consistently across instruments and participants is the presence of a strong relationship between a socio-constructivist and student-centred educational model and the implementation of ABM activities. This seems to align with what corresponds to a characterization of ABM activities by academic experts and research. Exploratory and meaning-making aspects also seem to be the ones most sought after by teachers in conducting these activities.

Moreover, among the expected outcomes of ABM activities implementation, we find strong agreement between the positions of academics and teachers implementing the activities: such activities offer deeper conceptual learning and have a lasting imprint in students' minds, they enhance mathematical visualization, promote student interest and succeed in engaging students.

Although these common treats, teachers are not aligned with researchers on the idea of inclusion: experts think this activity inclusive in a wide sense, for difficulties, for high achievers students, for different styles of learning (e.g. kinesthetic, visual learners). Among teachers, on the other side, many respondents think they are not adapt for high achievers or on the contrary for low achievers students.

The results show consistency between teachers' responses and the main constraints indicated by the academic experts. In particular, the academics pointed out that time pressure and coverage of curricular topics is one of the main factors inhibiting teachers from proposing these activities, which are also considered quite time- and energy-consuming for teachers. Indeed, one of the main limitations for proposing ABM activities pointed out by teachers is precisely the lack of available time. This is also the main reason why some teachers do not include these activities in their practice. In the follow-up interviews, we were able to get a better understanding of what teachers meant: on the one hand, the activities require a lot of time both in the classroom and in research and planning; on the other hand, with the time available for face-to-face instruction it is difficult to implement such activities, more time-consuming than traditional transmissive approaches, aiming to cover curricular contents. These statements highlight many subtexts, which can also be deduced from the analysis of other issues. Indeed, teachers do not seem to be so convinced that these activities bring results that are then reflected in standardized tests. Instead, according to academics, good results in these tests are the main goal of schools, which tend to measure themselves against NAPLAN (in Australia) / INVALSI (in Italy) assessments and international rankings (TIMSS, OECD investigations). It is therefore clear that the proposal is perceived by many teachers as ancillary to the planning and goals they are called upon to achieve. The other main difficulties identified are problems with classroom management and resource availability, which are factors also pointed out by academics. This relates both to affordability and, again, to availability of materials and resources, without spending a lot of time looking for them. While the context may therefore limit the implementation of ABM activities, the beliefs of teachers in prioritizing more traditional teaching methods geared toward content transmission, in order to cover the curriculum, should not be underestimated.

As highlighted by academic Expert 4, teachers often refer to the lack of time and available resources when new teaching strategies are proposed to them. Usually, these arguments actually hide the belief that what you are proposing is an extra activity from their programming:

> Another belief, I think, which is related to practice, and this is - this comes into play any time when you're proposing something to teachers that is different or new that you want them to try, they will tend to treat this as something "extra", something more, something additional, and in addition to what they're already doing. So, then, you're up against time: "I don't have time to do this, I have a curriculum to cover, there's exams" and so on. So, [could be relevant] helping teachers understand that no, what I'm proposing can actually replace some of the things you are doing, without losing anything. So, by doing things this way instead of some other things you are currently doing, you will still be able to achieve - or the students will still be able to achieve - the learning outcomes in the curriculum. Now, that is not something teachers might find easy to accept, right? - at the start. (Expert 4, p.34)

One dimension that was not initially given much consideration, but which seems to heavily impact the ABM activities implementation, is the school's teaching culture. Both the academics' and teachers' statements (e.g., in the questionnaire, through the brief indications expressed in the *Other* alternative within the items, but also in some of the follow-up answers), emphasized how this factor heavily inhibits implementation. This affects both fellow teachers and the entire school staff as well as the students themselves, but also their parents.

Limitation and further steps

Limitation

First of all, in our survey looking at the implementation of ABM activities, and in particular at the perspective of teachers, there is a heavy *bias* within the sample of teachers. Despite our efforts to try and track down teachers from outside the circuits that are close to the world of research and universities, there is a limitation in the research design that resulted in the selection of teachers who participated. Indeed, as participation in the research was voluntary, certainly the teachers who took part in the research tended to be interested in the subject matter. We therefore expect them to present a more general openness towards the proposal of these practices. Therefore, all results should be considered assuming that the research exhibits this imbalance.

Secondly, the number of participants reached in Australia are not remotely comparable to those contacted in Italy. Therefore, the possibility of making any comparison between the results is ruled out. Therefore, from the information gained we are merely able to draw hypotheses on the observed trends.

Furthermore, field research was precluded by the fact that the research project could be conducted only at distance, due to the pandemic emergency that broke out a few months after the start of the doctoral project. This contingency precluded the possibility of conducting case-studies observing teaching practices in both Italian and Australian classrooms. Therefore, the investigation could only be confined to teachers' statements, which are often far removed from daily teaching practice.

In addition, we attempted to briefly describe some features of Australian teaching culture possibly influencing the topic under investigation. However, further study would have been necessary to give voice to the complexity that takes into account historical, political and cultural roots.

Further steps

Teachers' involvement in the research revealed the presence of a wide variety of proposals that are currently implemented in schools, even very different from each other in terms of tools/materials involved, the instructional orientation in teaching strategies, and the content knowledge concerned. It could be relevant to conduct some case-studies, observing in classroom the actual implementation of these activities and, in particular, to also analyse students' perspectives on the ABM activities. Indeed, students are the only key actors excluded from the research landscape we considered, while also being the "end users" of the research findings implementation.

Moreover, many teachers emphasized that participation was a precious opportunity to reflect on their own teaching practices and openness to other teaching methods. Furthermore, they strongly expressed interest in receiving support, collaboration, and resources to better implement ABM activities in their daily practice. Therefore, it would be an interesting prospect to deepen the dialogue among the interested teachers (both the ones that are already carrying out the implementation of ABM activities and the ones that willing to do it), offering formative courses to put in practice the suggestions provided by the researchers, trying to find answers to some of the needs expressed by teachers.

Finally, recurring and possibly relevant further data emerged from the survey, both in the follow-up interviews of teachers and from the analysis of the short indications provided in the alternative *Other* within the questionnaire items. These indications represent possible future research directions that have not yet been explored.

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